ABSTRACT

In this dissertation the first problem of Stoke's for the rotating flow of third grade fluid will be considered. A method known as Lie group method which reduces the system of nonlinear partial differential equations to a system of ordinary differential equations on the basis of the underlying symmetry structure has been adopted. The Lie method is quite useful in reducing a complex system to an easy-to-handle system of ordinary differential equation. As the governing equations describing the fluid motions are highly complex and nonlinear in nature. The Lie group method seems to be an appropriate choice to handle these nonlinear equations. In this dissertation the Lie group structure for problem will be found under discussion and thereby using the Lie symmetries to obtain the reductions. Further,a series type solution for the problem considered have been obtained .

ABSTRAK

Dalam tesis ini, masalah pertama Stoke untuk memainkan aliran bendalir kelas ketiga akan dipertimbangkan. Kaedah yang dikenali sebagai kaedah kumpulan Lie yang menurunkan sistem persamaan pembezaan separa tak linear kepada sistem persamaan pembezaan biasa berdasarkan struktur simetri pendasar telah digunakan. Kaedah Lie ini amat berguna dalam menurunkan suatu sistem yang kompleks kepada sistem persamaan pembezaan biasa yang ringkas. Disebabkan, persamaan yang menggambarkan gerakan bendalir adalah sangat kompleks dan tak linear, keadah kumpulan Lie menjadi pilihan yang sesuai untuk menangani persamaan tak linear. Dalam disertasi ini, struktur kumpulan Lie untuk masalah yang dibincangkan telah dicari dan penggunakan simetri Lie untuk mendapatkan pengurangan diperolehi. Seterusnya, hubungan jenis siri untuk masalah yang dipertimbangkan telah didapatkan.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Problem

The first attempt to apply group-theoretic method (Lie group method) to handle differential equations was done by the renowned mathematician Sophus Lie. Lie made a crucial discovery, which was unified by meaning of group theory. During 1870-74, Lie investigated the role of a general transformation theory in classical integration methods and developed his ideas on what he called *finite continuous groups of transformations*. On that time he recognized the importance of the new notion in geometry and particularly in theory of partial differential equation. He also introduced the concept of infinitesimal contact transformation and developed new integration methods for partial differential equations of first order [1].

During the 20th centuries much progress has been made towards solving ordinary/partial differential equations through Lie group method. The few noteworthy contributions are: Vessiot [2] and Dickson [3] investigated the significance of applying transformation groups for solving differential equations. Vessiot worked with finite groups, whereas Dickson worked with infinitesimal groups. Cohen [4], concentrated on the application of one-parameter transformation groups to ordinary differential equations and first order partial differential equations while Eisenhart [5], considered multi parameter groups as well. Birkhoff [6] hypothesized that a reduction in the number of

1.3 Objectives of the Research

The objectives of this research are:

i. To understand the basic concept of Lie group analysis.

ii. To determine the Lie Algebra structure for the the first problem of Stoke's for rotating flow of third grade fluid.

iii. To perform the reduction in order by using the symmetries of the problem for defining exact solution.

1.4 Scope of the Research

In this dissertation some basic concepts of Lie group method will be introduced. Through the Lie group method for solving the first problem of Stoke's for rotating flow of third grade fluid, first the infinitesimals to write full one-parameter group of transformations (infinitesimal transformation group) will be found. From there with the help of the infinitesimal transformation group the generators and the Lie algebra of the nonlinear governing equations will be obtained. Then with applying symmetries the reduction will be performed and at the end the similarity solutions for the problem will be found.

1.5 Significance of the Research

The basic governing equations describing the rotating fluid phenomenon are non-linear in nature. The perturbation technique is the one which is widely used by physicists and engineers to handle these types of non-linear physical phenomena. Most of the time, we obtain many interesting and important results by utilizing this technique. However, the perturbation methods have their own limitations such as: all perturbation techniques are based on small or large parameters so that at least one unknown must be expressed in a series

(1.7) and (1.8), and after neglecting pressure gradient, (1.2) in the combined form takes the following form:

$$\frac{\partial F}{\partial t} + 2i\Omega F = \nu \frac{\partial^2 F}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 F}{\partial z^2 \partial t} + \frac{2\beta_3}{\rho} \frac{\partial}{\partial z} \left\{ \left(\frac{\partial F}{\partial z} \right)^2 \overline{\frac{\partial F}{\partial z}} \right\}, \tag{1.9}$$

where

$$F = u + iv$$

and

$$\bar{F} = u - iv. \tag{1.10}$$

It is convenient to write the problem in dimensionless variables. For that we introduce the following variables:

$$x = \frac{U_0}{\nu} z, \qquad \tau = \frac{{U_0}^2}{\nu} t, \qquad f = \frac{F}{U_0}, \qquad C = \frac{{U_0}^2}{\nu} \Omega. \tag{1.11}$$

The problem becomes

$$\frac{\partial f}{\partial \tau} + 2iCf = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^3 f}{\partial x^2 \partial \tau} + 2b \frac{\partial}{\partial x} \left\{ \left(\frac{\partial f}{\partial x} \right)^2 \overline{\frac{\partial f}{\partial x}} \right\}, \tag{1.12}$$

in which

$$a = \frac{\alpha_1 {U_0}^2}{\rho \nu^2}, \qquad b = \frac{\beta_1 {U_0}^4}{\rho \nu^3}. \tag{1.13}$$

The equation (1.12) in real and imaginary part can be written as:

$$\frac{\partial u}{\partial \tau} - 2Cv - \frac{\partial^2 u}{\partial x^2} - a \frac{\partial^3 u}{\partial x^2 \partial \tau} - 6b \left(\frac{\partial u}{\partial x} \right)^2 \left(\frac{\partial^2 u}{\partial x^2} \right) - 2b \left(\frac{\partial v}{\partial x} \right)^2 \left(\frac{\partial^2 u}{\partial x^2} \right)$$
$$-4b \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial v}{\partial x} \right) \left(\frac{\partial^2 v}{\partial x^2} \right) = 0, \tag{1.14}$$

$$\frac{\partial v}{\partial \tau} + 2Cu - \frac{\partial^2 v}{\partial x^2} - a \frac{\partial^3 v}{\partial x^2 \partial \tau} - 6b \left(\frac{\partial v}{\partial x} \right)^2 \left(\frac{\partial^2 v}{\partial x^2} \right) - 2b \left(\frac{\partial u}{\partial x} \right)^2 \left(\frac{\partial^2 v}{\partial x^2} \right)$$
$$-4b \left(\frac{\partial v}{\partial x} \right) \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial^2 u}{\partial x^2} \right) = 0. \tag{1.15}$$

1.2 Derivation of the Problem

Consider an incompressible third grade fluid occupying the space $Z > 0$. The plate at $Z = 0$ is moved suddenly with a constant velocity for $t > 0$. Both the fluid and plate are in a solid rotation. Initially the fluid and plate are at rest. The laws which govern the flow

$$div\mathbf{V} = \mathbf{0}. \tag{1.1}$$

$$\rho\Big[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\mathbf{V} + 2\boldsymbol{\Omega}\times\mathbf{V} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\Big] = -\nabla p + div\mathbf{T}, \tag{1.2}$$

in which $\mathbf{V}$ is the velocity, ρ is the density, t is the time, p is hydrostatic pressure, $\mathbf{T}$ is the extra stress tensor, $\boldsymbol{\Omega}$ is the constant angular velocity and $\mathbf{r}$ is the radial coordinate with $\mathbf{r}^2 = x^2 + y^2$. The extra stress tensor $\mathbf{T}$ in a third grade fluid is

$$\mathbf{T} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{1.3}$$

Here μ is the dynamic viscosity, α_i $(i = 1, 2)$, and β_i $(i = 1 - 3)$ are the material constants. The kinematical tensor $\mathbf{A}_n$ are

$$\mathbf{A}_1 = \boldsymbol{\nabla}\boldsymbol{V} + (\boldsymbol{\nabla}\boldsymbol{V})^2, \tag{1.4}$$

$$\mathbf{A}_n = \Big(\frac{\partial}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{V})\Big)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\boldsymbol{\nabla}\mathbf{V}) + (\boldsymbol{\nabla}\mathbf{V})^T\mathbf{A}_{n-1}, n = 2, 3, ... \tag{1.5}$$

The thermodynamics of the fluid requires that [31]

$$\begin{aligned} &\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad |\ \alpha_1 + \alpha_2\ | \geq \sqrt{24\mu\beta_3} \\ &\beta_1 = \beta_2 = 0, \qquad \beta_3 \geq 0. \end{aligned} \tag{1.6}$$

Therefore, Eq. (1.3) can be written as

$$\mathbf{T} = [\mu + \beta_3(tr\mathbf{A}_1^2)]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{1.7}$$

The velocity field $\mathbf{V}$ for the present flow problem is given as

$$\mathbf{V} = [u(z, t), v(z, t), w(z, t)], \tag{1.8}$$

which together with the incompressibility condition yields $w = 0$ (u, v and w are the velocity component in x, y, z directions, respectively). In view of equations independent variables in a partial differential equation could be attained by using transformation groups. Morgan [7] showed that this was indeed the case by proving that a one-parameter transformation group leads to a reduction by one in the number of independent variables. Ovsiannikov [8] outlined a systematic method for obtaining the most general infinitesimal equations invariant. A discussion of the method and examples of its application may be found in [9-11]. In Birkhoff [6] it is shown that any reduction in the number of independent parameters obtainable through dimensional analysis may also be obtained using a group of stretching transformations.

If we start with a system of partial differential equation with two independent variables and apply a one-parameter infinitesimal transformation group to it, we obtain a system of ordinary differential equations which, in general, is easier to solve than the original system. For this reason, most of the equations analyzed thus far have been partial differential equations with two independent variables. The first person to use the most general infinitesimal transformation group to find invariant solutions of partial differential equation was Ovsiannikov [8] who studied the nonlinear diffusion equation $[f(u)u_x]_x = u_t$. For past two decades, a lot of researchers working in Mathematical Physics, Fluid Mechanics and Engineering fields have utilized Lie group method in their models to obtain the exact analytical solutions [12-30].

In this dissertation, Lie group method will be used to determine the Lie group structure for the first problem of Stoke's for rotating flow of third grade fluid. Further by using Lie symmetries, we will obtain the reduction of nonlinear system of partial differential equations to a system of ordinary differential equations and thereby obtain the solutions of the problem.

of small parameters. Unfortunately, not every non-linear differential equation has this kind of structure. Even if there exists such a small parameter, the results given by perturbation methods are valid, in most cases, only for the small values of the parameter. Mostly, the simplified linear equations have different properties from the original non-linear differential equation, and sometimes some initial or boundary conditions are superfluous for the simplified linear equations. As a result, the corresponding initial approximations are perhaps far from exact. Clearly, these limitations of perturbation techniques arise from the small parameter assumption. So the significant part is to tackle the governing nonlinear equations which govern the flow with such a method which does not require these limitations. For that first we will use Lie group method which does not require small parameter assumptions at all. Second the algebraic structure which we obtain here will provide us the mechanism to search for other solutions since its character is inferred from the basic equations.

1.6 Overview

In Chapter 1 history of solving non-linear equations have been stated. Regarding to this history, some mathematicians have investigated different methods for solving these kind of problems which a brief state of these methods have been provided.In particular, purpose and importance of this study have been considered through the last Sections. Some literatures of this study have been provided trough the Chapter 2. In Chapter 3, some basic definitions have been provided and the method for solving these kind of problem have been show through an example. The similarity solution for problem have been explained through the Chapter 4. In this Chapter, the problem have been reduced to an ordinary differential equation by employing Lie group method, and since the resulting ordinary differential equation was non-linear with variable coefficient a series type solution have been considered to approximate it. A discussion of the results along with their graphs has been provided through the last section.